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A numerical study of the descent of a vortex pair in a stably stratified atmosphere

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Numerical methods are used to investigate the motion of a horizontal vortex pair through a stably stratified atmosphere. The vortices carry with them a mass of fluid whose density differs from that of the air through which it descends, and the surface of this accompanying fluid becomes a vortex sheet, which is modelled by a set of discrete line vortices.

It is shown that, at first, the vortex pair slows down with the shape of the envelope of the accompanying fluid remaining constant. Later, vorticity concentrates at the rear, initiating detrainment and causing a downward acceleration of the vortex pair. Throughout the motion, the vortices approach each other.

1. Introduction

The trailing vortices formed behind an aircraft descend vertically through the atmosphere, carrying with them a mass of fluid whose density, neglecting the engine exhaust, is that of the air through which the aircraft is flying. As this mass of fluid descends, it will, in general, encounter fluid of different density and therefore buoyancy forces will affect its motion. To study the effect of buoyancy forces on the descent, the more tractable situation of the unsteady two-dimensional motion of a pair of rectilinear line vortices is considered, where the time is proportional to the distance behind the aircraft.

Now in a uniform atmosphere, the descent velocity of the vortex pair is given by

$$V = \kappa/4\pi R, \tag{1.1}$$

where κ and $-\kappa$ are the circulations about the right-hand and left-hand vortices, respectively, and $2R$ is their separation. Fluid within the contour C is carried along with the vortices, where the shape of C is given by (Lamb 1932, p. 221)

$$\frac{x}{R} + \log\frac{(x-R)^2+y^2}{(x+R)^2+y^2} = 0. \tag{1.2}$$

Here x and y are co-ordinates with respect to axes moving with the vortices and with origin midway between them, Ox being horizontal. The curve C plays a crucial role in the theories to be described because in a stratified atmosphere it is the boundary between fluids of different densities. Of course, as the motion proceeds, C will be distorted from the form (1.2) by density effects, and one object of this study is to find the modification to C.

Scorer & Davenport (1970) were the first to consider the motion of a vortex pair in a stratified atmosphere. They allowed for buoyancy effects by equating the rate of change of the impulse of the vortex pair to the total buoyancy force, neglecting the change in impulse due to vorticity generated baroclinically on the envelope C. They predicted an increase in the downward velocity of the vortex pair and a concomitant decrease in their separation. Clearly, then, fluid would have to be detrained from the upper surface of C.

Saffman (1972) obtained an approximate solution to the problem by supposing that the shape of C and the distance between the vortices remain constant. He determined a velocity potential which satisfied the kinematic conditions at the boundary exactly, but satisfied the pressure condition at the boundary only approximately. The surface C is a vortex sheet, representing the vorticity generated baroclinically, and hence this effect is allowed for in Saffman's work. With no detrainment, Saffman found that the vortex pair slowed down and eventually reversed its motion, subsequently oscillating between two levels.

Both Saffman and Scorer & Davenport neglected viscosity and the vorticity generated baroclinically away from the envelope C, these two approximations being made in this paper also. The object of the work described here is to try to decide, by a numerical approach, which of these approximate theories is closer to the truth.

More recently, support for Scorer's theory has been furnished by Crow (1974). He allowed for the baroclinic generation of vorticity on C in an approximate fashion and concluded that detrainment would have to occur.

In § 2, a description of the physical model is given together with an account of how the equations governing the motion were written in a form accessible to numerical analysis. A check on the method of integration is described in § 3, and a discussion of the results obtained from numerical calculations follows in § 4.

2. The motion of a pair of line vortices in a stratified atmosphere

The problem considered is that of a pair of line vortices descending vertically through an atmosphere with density stratification defined by

$$\rho = \rho_0(1-\beta Y), \tag{2.1}$$

where ρ_0 is the density at the fixed height $Y = 0$, the flight level of the aircraft, and $R\beta \ll 1$. The fluid moving with the vortex pair is supposed to be of constant density ρ_c, where

$$\rho_c = \rho_0(1-\beta H). \tag{2.2}$$

Thus the density of the air moving with the pair need not be equal to that of the ambient atmosphere, to allow for the effect of the engine exhaust.

If a frame of axes (x, y) moving with and centred on the vortex pair is introduced, as in § 1,

$$Y = y - I(t), \tag{2.3}$$

where

$$I(t) = \int_0^t V(t)\,dt \tag{2.4}$$

and $V(t)$ is the downward velocity of the vortex pair.

Bjerknes' theorem states that

$$\frac{d\Gamma}{dt} = \int_{A'}\int \nabla\frac{1}{\rho}\times\nabla p \, . \, d\mathbf{A}', \tag{2.5}$$

where $\Gamma(t)$ is the circulation around the material circuit C' enclosing the area A'. The fluid accelerations near C can be estimated from the solution without density differences. For the cases of interest, they prove to be much smaller than the gravitational acceleration, so that it is legitimate to employ the hydrostatic approximation, as did Scorer & Davenport (1970) and Saffman (1972). Then ∇p can be replaced by its static value and (2.5) becomes

$$\frac{d\Gamma}{dt} = -g\int_{A'}\int \frac{1}{\rho}\frac{d\rho}{dx}\,dx\,dy, \tag{2.6}$$

where g is the acceleration due to gravity.

The integral in (2.6) is zero for any area A inside C, and if the baroclinic generation of vorticity outside C is neglected, a second approximation made by Scorer & Davenport and by Saffman, it is seen that vorticity is created only on the envelope C. In particular, it is clear that the strengths of the two main vortices remain constant. Hence, as the motion proceeds, C becomes a vortex sheet.

To discretize the problem, C is represented by a finite number $2M$ of fluid particles, whose initial spacing is uniform. As the motion proceeds, each fluid particle is made to become a point vortex whose instantaneous strength is determined by Bjerknes' theorem. Vortex lines are not material in a stratified fluid and so the discretization familiar in aerodynamics, where vortex sheets are replaced by point vortices of constant strength, is not useful here. The symmetry of the problem enables the motion on the right-hand side only to be followed.

Suppose that P_j, whose co-ordinates are (x_j, y_j), is the position of the jth vortex and let Γ_j be its instantaneous strength. The objective of this analysis is to derive from the Boussinesq form of Bjerknes' theorem an equation for $d\Gamma_j/dt$. If the vortices are closely spaced on C, the portion of C with end points P_{j-1} and P_{j+1} can be regarded as a straight line. Let L_j be the midpoint of $P_{j-1}P_j$ and let R_j be the midpoint of P_jP_{j+1} and apply Bjerknes' theorem to a rectangle A_j whose sides are parallel to $P_{j-1}P_jP_{j+1}$ and whose ends pass through L_j and R_j. To calculate the integral (2.6), take new axes Ox' and Oy', where O is at P_j and Ox' lies along P_jP_{j+1}. Then, neglecting the variation of ρ parallel to $P_{j-1}P_{j+1}$,

$$\rho(y') = \rho_j + (\rho_j - \rho_c)H(y'),$$

where ρ_j is the value of ρ at P_j. Thus,

$$\frac{d\Gamma_j}{dt} = g\sin\theta_j \iint_{A_j} \frac{\partial}{\partial y'}\log\rho\,dx'\,dy',$$

where θ_j is the inclination of the line $P_{j-1}P_jP_{j+1}$ above the horizontal. Performing the integration gives

$$d\Gamma_j/dt = g\delta_j\sin\theta_j\log(\rho_c/\rho_j),$$

where δ_j is the length of L_jR_j. Finally, ρ_j is obtained from (2.1)–(2.4) to give

$$\frac{d\Gamma_j}{dt} = -g\delta_j\sin\theta_j\log\left[1+\frac{\beta}{1-\beta H}(H+I(t)-y_j)\right]. \tag{2.7}$$

The quantity $\delta_j \sin\theta_j$ is readily calculated in terms of the co-ordinates of the point vortices, so that (2.7) provides an equation for the rate of change of the strength of each vortex, given the configuration. Equations for the rate of change of the co-ordinates are obtained by calculating the velocity of any point vortex in terms of the positions and instantaneous strengths of the other vortices. Symmetry is imposed, so that, to a point vortex of strength Γ_j at (x_j, y_j) on the right-hand portion of C, there corresponds a vortex of strength $-\Gamma_j$ at $(-x_j, y_j)$ on the left-hand portion.

The resulting system of ordinary differential equations was integrated forwards in time and the positions of the main vortex pair, the location C of the density discontinuity and the distribution of baroclinically generated vorticity on C were deduced. Fourth-order Runge–Kutta integration was used.

Both Scorer & Davenport and Saffman neglected the variation in density along C, therefore in order to compare the numerical results with their theories, ρ_j/ρ_c was given the value $1+\beta(1-\beta H)^{-1}(H+I(t))$. However calculating ρ_j using its true local value instead of the value at the pair level had little effect on the results.

3. Check on the method of integration

To check the program and test its accuracy, it was used to predict a flow known analytically. If the motion starts from rest, the amplitude, in linear theory, of a sinusoidal wave on a horizontal interface between two fluids of different densities $\rho_1 > \rho_2$, with the heavier fluid uppermost, is proportional to $\cosh \sigma t$, where

$$\sigma^2 = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} gk. \tag{3.1}$$

Here, k is the wavenumber of the disturbance, g is the acceleration due to gravity and t is the time measured from the instant at which the fluid is at rest.

Using the discrete-vortex approximation, M point vortices of initial strength zero were placed along one wavelength of the initial disturbance and were allowed to increase in strength according to Bjerknes' theorem. The other waves were allowed for by remarking that, corresponding to the vortex of strength C_j at (x_j, y_j), there must be a vortex of strength at $(x_j \pm 2n\pi/k,\ y_j)$ for $n = 1, 2, 3, \ldots$. The contribution of all the vortices with a given j at the ith vortex could be found analytically, as was done by Rosenhead (1931). Thus the motion of the M vortices was followed by marching forward in time, as before.

The results obtained are displayed in figure 1, where m, the magnification of the initial wave amplitude, is plotted against the non-dimensional time $T = \sigma t$. The values of the initial amplitude and wavenumber used in the program were $0{\cdot}2$ m and $0{\cdot}1\ \text{m}^{-1}$, respectively, and because the Boussinesq approximation is valid in the present situation only if $(\rho_1-\rho_2)/(\rho_1+\rho_2) \ll 1$, this ratio was given the value 10^{-3}.

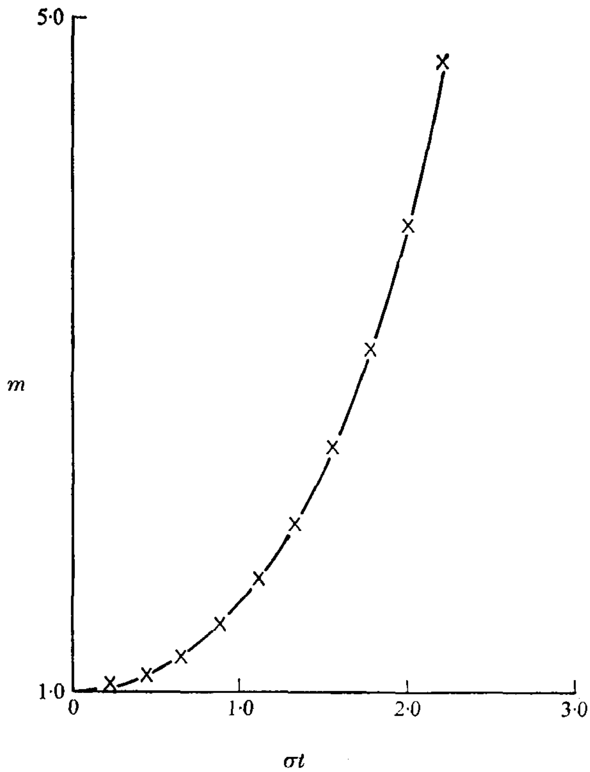

FIGURE 1. Comparison between the results of the discrete vortex approximation and the analytical solution for the amplitude of a sinusoidal wave on the horizontal interface between two fluids of slightly different densities. ——, cosh T.

4. Results

It was first necessary to determine the value of $2M$, the number of fluid particles representing the vortex sheet, and the time step.

The calculations were done with $M = 30$, 40, 50 and 98. The differences in the values obtained for the downward velocity and the vertical and horizontal displacements of the vortex pair proved to be extremely small, the largest discrepancy being less than 0·01 %. Thus it was inferred that $M = 30$ gave a sufficiently good resolution of the flow for the above quantities to be determined. However, the value $M = 98$ was used in some runs to enable the shape of C to be determined accurately.

The time step used was $0{\cdot}015\,T_0$, where $T_0 = 4\pi R^2/\kappa$ is the time required for the pair to descend a distance equal to half their separation in the unstratified case. Calculations with shorter time steps produced the same results.

Before considering the results themselves, it is helpful to consider how they can depend on the imposed parameters. Crow (1974) has identified the dimensionless groups involved and (for the case $H = 0$) his results show that the descent velocity is

$$(\kappa/4\pi R)\, U(Nt, NT_0, (\kappa/4\pi)^2 (gR^3)^{-1}),$$

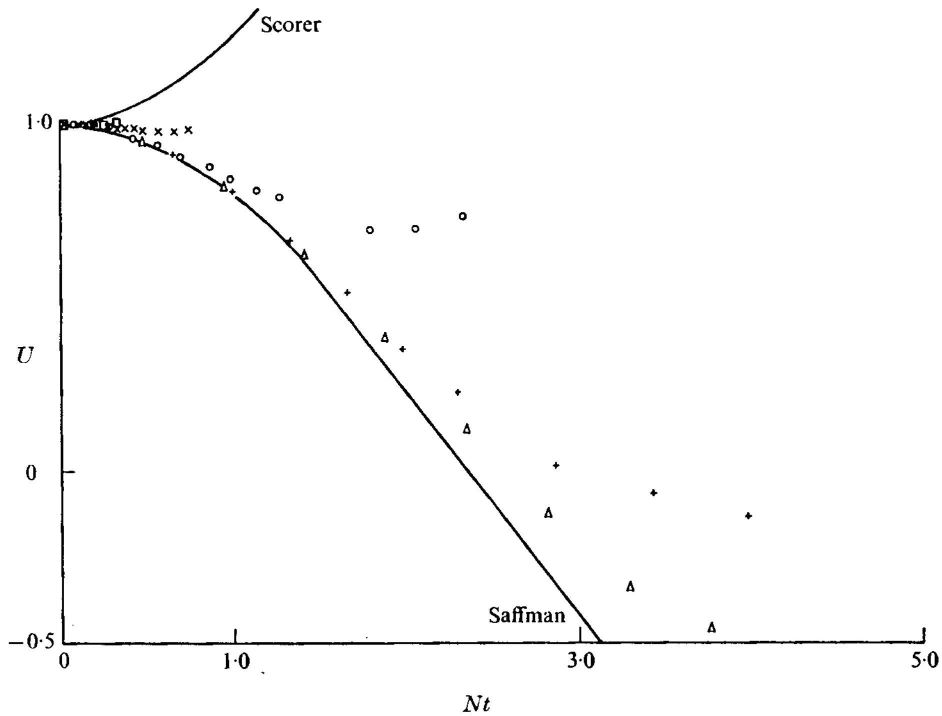

FIGURE 2. Comparison of the numerical results with the theoretical results of Scorer & Davenport (1970) and Saffman (1972) for U. □, $\beta = 10^{-5}\,\mathrm{m}^{-1}$; ×, $\beta = 10^{-4}\,\mathrm{m}^{-1}$; ○, $\beta = 10^{-3}\,\mathrm{m}^{-1}$; +, $\beta = 5\times10^{-3}\,\mathrm{m}^{-1}$; △, $\beta = 10^{-2}\,\mathrm{m}^{-1}$.

where $N = (g\beta)^{\frac{1}{2}}$ is the Brunt–Väisälä frequency. Crow points out that in making the hydrostatic approximation one is treating $(\kappa/4\pi)^2\,(gR^3)^{-1}$ as a small quantity, because this group is the ratio of the fluid accelerations $(\kappa/4\pi)^2\,R^{-3}$ caused by the vortex pair to that due to gravity, g. However, U cannot depend on $(\kappa/4\pi)^2\,(gR^3)^{-1}$ because β and g are inextricably linked. In all the calculations, $\kappa/4\pi$ had the value $60\,\mathrm{m}^2\,\mathrm{s}^{-1}$ and R had the value 20 m, which are typical of large transport aircraft. Thus the parameter $(\kappa/4\pi)^2\,(gR^3)^{-1}$ had the value 0·046. H was given the value zero in the calculations to be described here.

It may be noted at this point that Scorer & Davenport predict a descent velocity

$$(\kappa/4\pi R)\cosh(0{\cdot}67\,Nt)$$

while Saffman predicts a descent velocity

$$(\kappa/4\pi R)\cos(0{\cdot}67\,Nt).$$

Calculations were done with $\beta = 10^{-5}$, 10^{-4}, 10^{-3}, 5×10^{-3} and $10^{-2}\,\mathrm{m}^{-1}$. The range 10^{-5}–$10^{-4}\,\mathrm{m}^{-1}$ is typical of the real atmosphere (Scorer & Davenport 1970).

The results for the descent velocity are shown in figure 2 with Scorer & Davenport's and Saffman's results shown for comparison. It can be seen that Saffman's prediction that the pair is decelerated is supported by the present results.

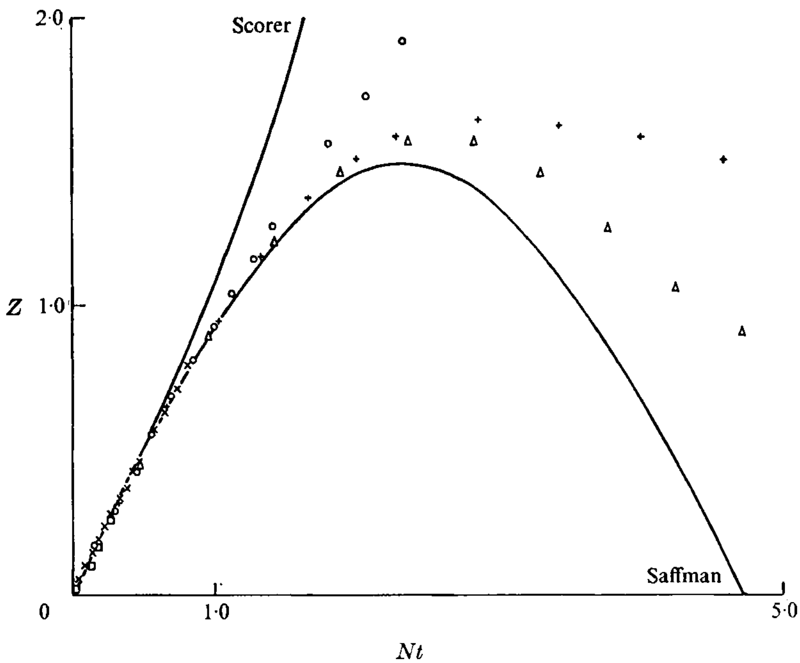

FIGURE 3. Comparison of the numerical results with the theoretical results of Scorer & Davenport (1970) and Saffman (1972) for Z. Symbols as in figure 2.

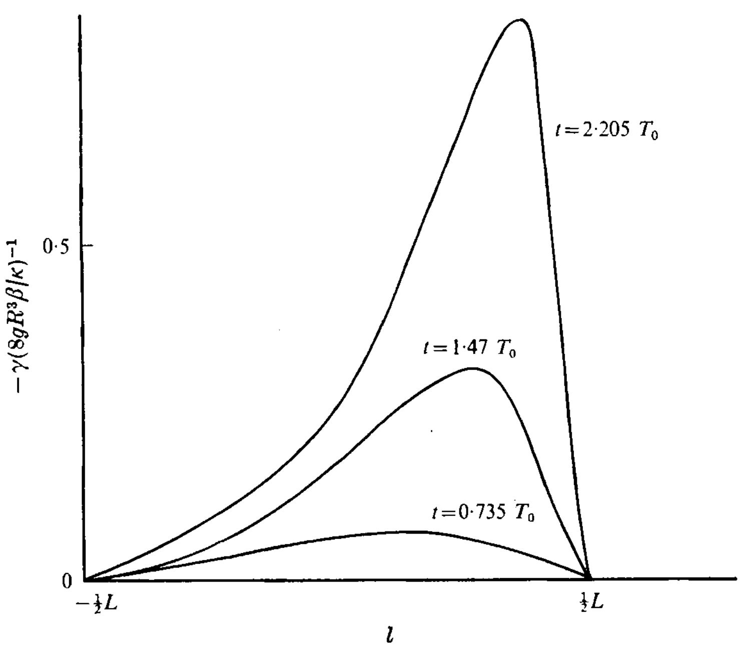

FIGURE 4. Distribution of vorticity on the right-hand portion of C determined from the discrete vortex approximation. γ is defined by Crow (1974).

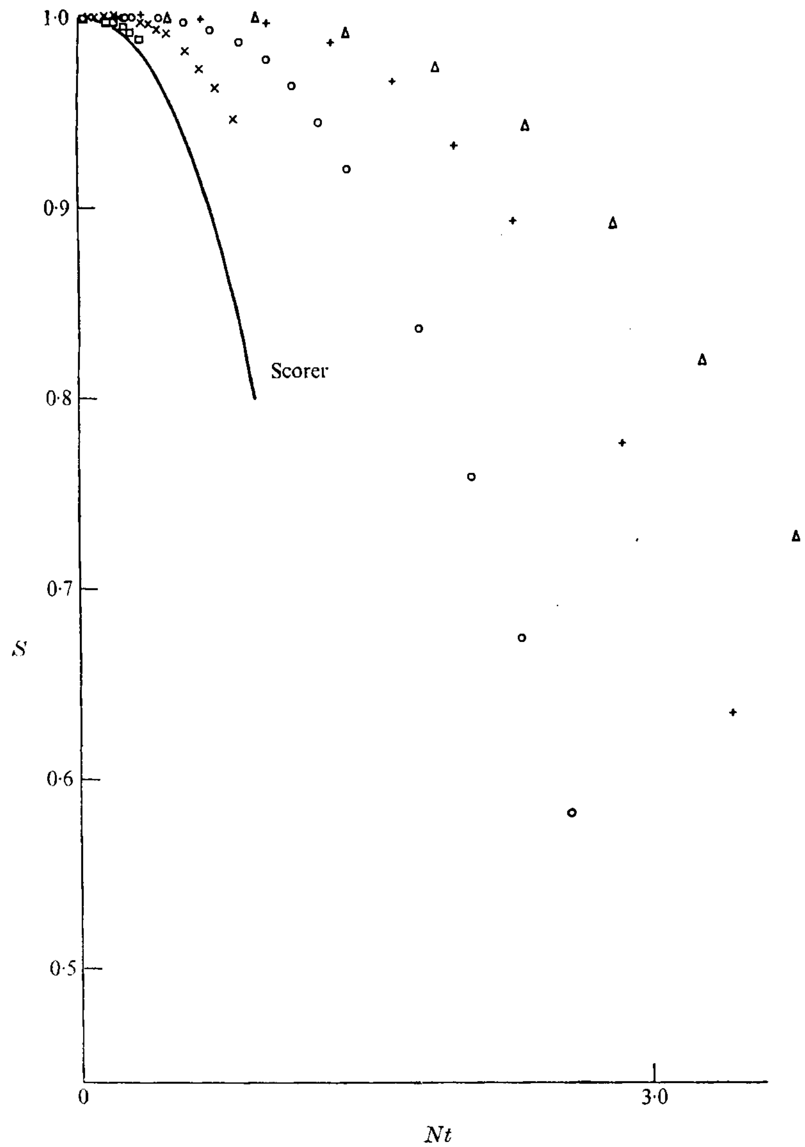

FIGURE 5. Comparison between the numerical results and the theoretical results of Scorer & Davenport (1970) for S. Symbols as in figure 2.

However, Saffman's theory underestimates the descent velocity, which indeed eventually starts to increase. However, this eventual acceleration is not in agreement with Scorer & Davenport's predictions.

A quantitative measure of the accuracy of Saffman's theory can be obtained by seeking the time $t_{1\cdot 1}$, at which $U/U(\text{Saffman}) = 1\cdot 1$. On dimensional grounds,

$$t_{1\cdot 1} = T_0 f(NT_0, (\kappa/4\pi)^2(gR^3)^{-1}),$$

and for the value of $(\kappa/4\pi)^2(gR^3)^{-1}$ used in the present calculations, it was found that

$$t_{1\cdot 1} = 1\cdot 25\, T_0(NT_0)^{-\frac{2}{3}}. \tag{4.1}$$

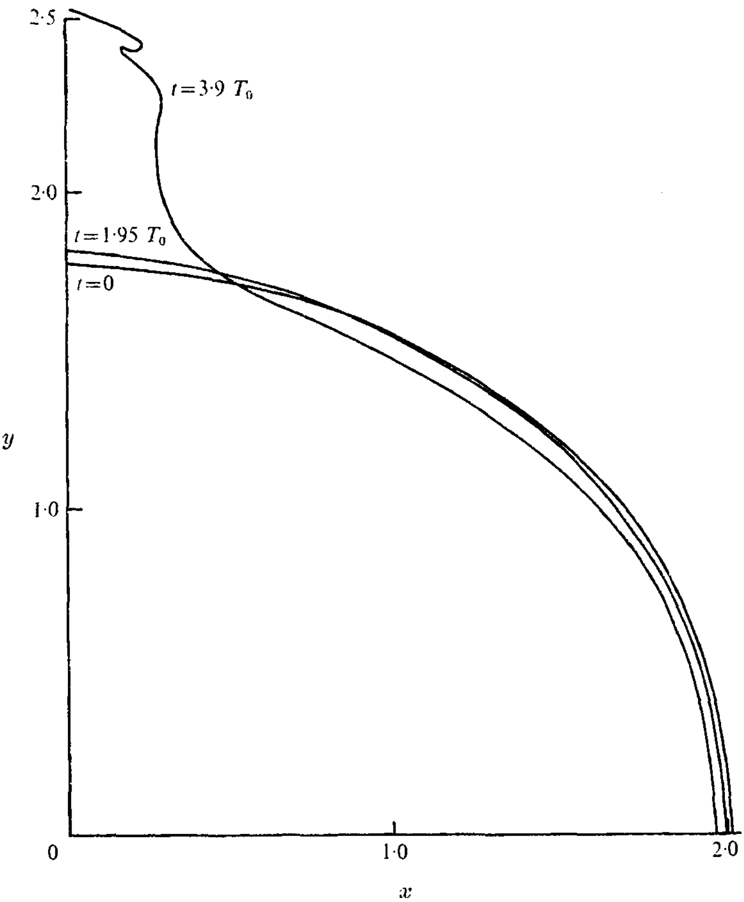

FIGURE 6. The shape of the upper right-hand portion of C at various times.

The results for the downward displacement are shown in figure 3, where

$$Z = (4\pi NR/\kappa)\, I(t).$$

Figure 4 shows the distribution of vorticity on C at different times. Vorticity of sign opposite to that in the main vortex concentrates near the upper stagnation point as t becomes greater than T_0. In figure 5 the vortex separation S is shown as a function of Nt. According to Saffman, it should be constant, but in fact the vortices move towards each other, the greatest decrease being less than 10 % for the calculation with $\beta = 10^{-4}\,\mathrm{m}^{-1}$. This can be interpreted as the effect of the vorticity of opposite sign, which is generated on C and which concentrates on the upper half of C at the larger times.

Figure 6 shows the upper half of C at various times when $\beta = 10^{-4}\,\mathrm{m}^{-1}$. This run was performed with $M = 98$ to give good resolution of the shape. It was not possible to use the program to resolve the flow for values of t larger than $4T_0$.

The program was modified to calculate the descent velocity of a vortex pair through an atmosphere of constant density different from that surrounding the vortex pair. The results obtained are compared with the motion predicted by Saffman in figure 7, the ratio of the two densities having the value 1·0022.

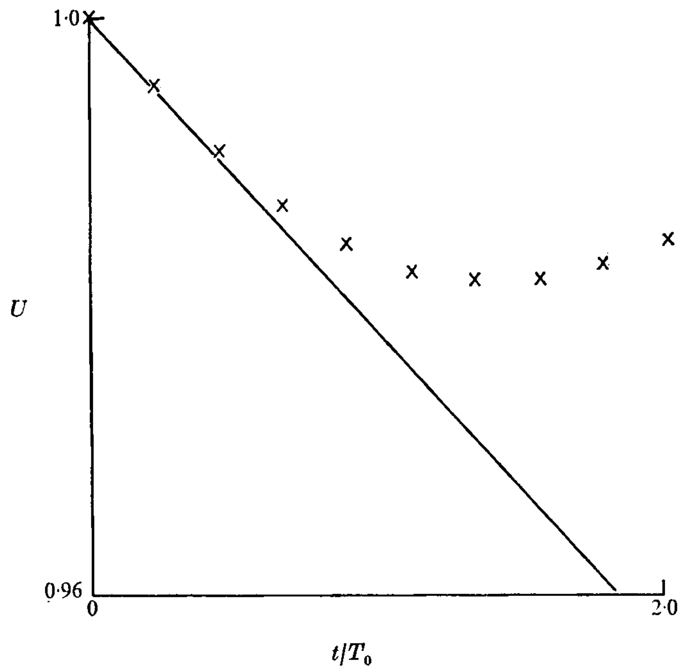

FIGURE 7. Comparison of the numerical results with the theoretical results of Saffman (1972) for U for a constant density ratio 1·0022 of the densities of the ambient fluid and the fluid moving with the vortex pair.

5. Discussion

The numerical results agree closely with the motion predicted by Saffman (1972) for small times, as suggested by (4.1). Disintegration of an aircraft vortex trail in an unstratified atmosphere occurs at approximately $6{\cdot}4\,T_0$ (Bisgood, Maltby & Dee 1971). Therefore Saffman's theory will adequately describe the effect of stratification if $\beta < 2 \times 10^{-5}\,\mathrm{m}^{-1}$ and will be adequate for more than half the disintegration time even if $\beta = 10^{-4}\,\mathrm{m}^{-1}$.

Crow's theory was based on the neglect of the time derivative in the vorticity equation and the neglect of the effect on this equation of the deformation of C, the boundary of the buoyant region. The vorticity equation is equivalent to

$$\frac{\partial \gamma}{\partial t} + \frac{\partial}{\partial l}(q(l)\gamma) = \frac{\Delta\rho}{\rho_0} g \sin\theta(l).$$

Here l is a curvilinear co-ordinate which describes the location on the interface C. γ is the strength of the vortex sheet on C and is defined by

$$\gamma = \int_{-\epsilon}^{+\epsilon} \zeta\, dn,$$

where n is the normal to C, ζ is the vorticity and ϵ is a small distance. $(\Delta\rho/\rho_0)g = c_1$ is constant in an atmosphere of constant density and $(\Delta\rho/\rho_0)g = c_2 t$ in a stratified atmosphere when t is small, c_2 being a constant. If distortion of C is neglected and the effect of the baroclinically generated vorticity on the velocity $q(l)$ is neglected, $q(l)$ is given by the unstratified theory.

These approximations enable the above equation to be solved using characteristics and this solution enables the accuracy of Crow's solution to be assessed. The solution found in this way is

$$\gamma(l,t) = \begin{cases} \dfrac{c_1}{q(l)}\,[y(l)-y(F(h(l)-t))], & \dfrac{\Delta\rho}{\rho_0}\,g = c_1, \\ \dfrac{c_2}{q(l)}\left[ty(l) - \displaystyle\int_{F(h(l)-t)}^{l} \frac{y(l')}{q(l')}\,dl'\right], & \dfrac{\Delta\rho}{\rho_0}\,g = c_2 t, \end{cases} \tag{5.1}$$

where
$$h(l) = \int_0^l \frac{du}{q(u)}$$

and F is the inverse function of h: F is unique because h is monotone increasing. The origin of l is chosen such that $-\frac{1}{2}L \leqslant l \leqslant \frac{1}{2}L$.

For small t, (5.1) becomes

$$\gamma(l,t) \sim \begin{cases} c_1 t \sin\theta(l), & (\Delta\rho/\rho_0)\,g = c_1, \\ \frac{1}{2}c_2 t^2 \sin\theta(l), & (\Delta\rho/\rho_0)\,g = tc_2. \end{cases}$$

Now $\sin\theta(l) = \sin\theta(-l)$ and therefore the distribution of baroclinically generated vorticity is symmetric in l at small times and shows no tendency to accumulate at the rear.

For large t, $F(h(l)-t) \sim F(-t)$ except at $l = \frac{1}{2}L$, where $h(l)$ becomes infinite. For $l \neq \frac{1}{2}L$, at large t, $F(h(l)-t) \to -\frac{1}{2}L$ and so

$$\gamma(l,t) \sim \frac{c_1}{q(l)}\,[y(l)+a], \quad \frac{\Delta\rho}{\rho_0}\,g = c_1,$$

where the bottom stagnation point is given by $y = -a$. This is Crow's expression when t is large. As $l \to \frac{1}{2}L$, $q(l) \sim \frac{1}{2}l$, $\sin\theta(l) \sim \frac{1}{2}L - l$ and $F(h(l)-t) \sim l - tq(l)$. So for any time t and $\frac{1}{2}L - l \ll 1$,

$$y(l) - y[F(h(l)-t)] \sim t(\tfrac{1}{2}L - l)^2$$

and therefore $\gamma(\frac{1}{2}L, t) = 0$ when $(\Delta\rho/\rho_0)\,g = c_1$. Similarly, for $(\Delta\rho/\rho_0)\,g = c_2 t$

$$\gamma(l,t) \sim \tfrac{1}{2}t^2(\tfrac{1}{2}L - l), \quad \tfrac{1}{2}L - l \ll 1.$$

Thus it appears that Crow's solution is valid only for times $\gg T_0$, and then not at the rear stagnation point.

Since $\gamma(0,t) > 0$ and $\gamma(\frac{1}{2}L, t) = 0$, a maximum value of γ may be anticipated for a value of $l > 0$. For an atmosphere of constant density, further insight into the magnitude and location of the peak value of γ may be gained. Inspection of (5.1) shows that a maximum value of γ must occur for a value of l greater than $F(\frac{1}{2}t)$. Suppose that this maximum occurs when $t = \tau(t)\,h(l)$, so that $0 < \tau(t) < 2$. Values of t in the above expression are considered for $l \sim \frac{1}{2}L$, when

$$q(l) \sim (\sqrt{3}/2T_0)\,(\tfrac{1}{2}L - l),$$

and therefore $h(l) \sim (-2T_0/\sqrt{3})\log(\frac{1}{2}L - l)$.

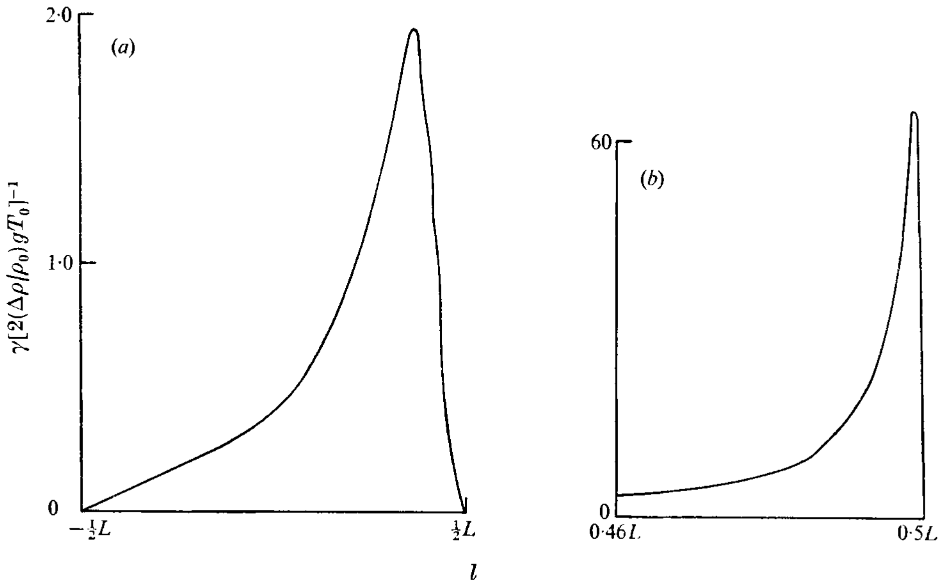

FIGURE 8. The vorticity distribution on C calculated numerically from the vorticity equation derived by Crow (1974) for an atmosphere of constant density. The non-dimensionalization of γ is suggested by (5.1). (a) $t = 2T_0$. (b) $t = 6T_0$.

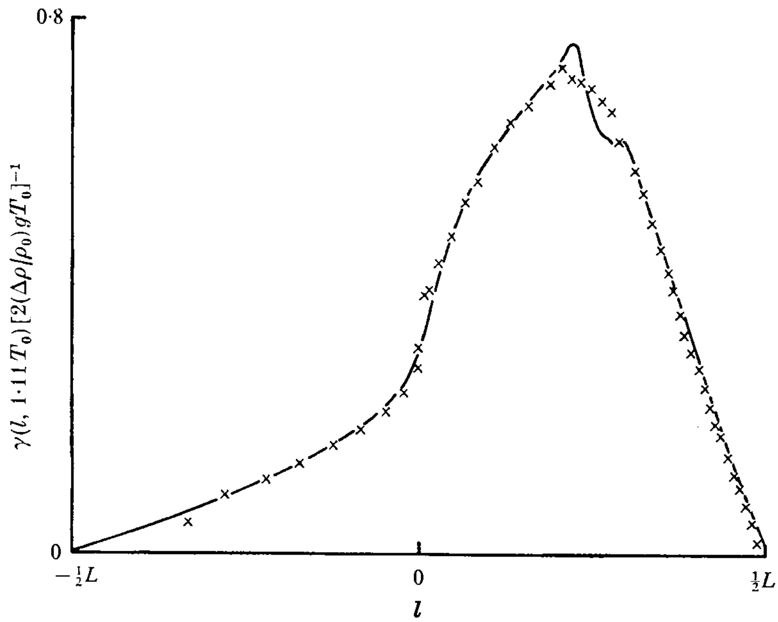

FIGURE 9. Comparison between the results of the discrete vortex approximation (crosses) and those derived from Crow's vorticity equation for the vorticity distribution on C at time $1{\cdot}1T_0$ for an atmosphere of constant density.

A maximum value γ_M of γ then occurs when

$$t = -\tau(t)\frac{2T_0}{\sqrt{3}}\log\left(\tfrac{1}{2}L-l\right),$$

or

$$\tfrac{1}{2}L-l = \exp\left[-t\sqrt{3}/2T_0\tau(t)\right],$$

and since

$$y(l)-y[F(h(l)-t)] \leqslant 2a,$$

$$\gamma_M \sim \exp\left[t\sqrt{3}/2T_0\tau(t)\right].$$

Thus there is a sharp peak close to the rear stagnation point, as may be seen in figure 8. $\gamma(l,t)$ was calculated numerically from (5.1) for several values of t for the constant-density case, and the results are displayed in figure 8. This enables a further check on the discrete vortex approximation program to be made because the above values of γ for $t = 1{\cdot}11T_0$, when C still has its original shape, can be compared with the corresponding values of γ determined from the results of the discrete vortex approximation. This comparison is shown in figure 9.

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